Interactions of Poisson Geometry, Lie Theory and Symmetry

PB-groupoids vs VB-groupoids

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In [1] we introduced principal bundle groupoids (PB-groupoids) as a natural counterpart of vector bundle groupoids (VB-groupoids) and discovered its rich interactions with Lie groupoids, principal bundles, and (strict) Lie 2-groupoids. We established a new correspondence between VB-groupoids and PB-groupoids expanding classical geometric constructions and providing new perspectives in the study of higher gauge theories. Additionally, our work includes illustrative examples with explicit formulas.

1. Lie groupoids and VB-groupoids

For proofs and more info about these objects see [2]. A Lie groupoid can be seen in different ways. This poster will see them as a manifold with a set of relations. To use the standard notation for these objects, the manifold will be called the base and the relations will be called arrows. Following this logic, we consider vector bundles (VB) over Lie groupoids (aka over manifolds and relations).

Definition 1.1. *A* **VB-groupoid of rank** (l,k) *is a commutative diagram*

$$\begin{array}{ccc} E_{\mathcal{G}} & \xrightarrow{\pi_{\mathcal{G}}} & \mathcal{G} \\ \tilde{t}\downarrow\downarrow\tilde{s} & & t\downarrow\downarrow s \\ E_M & \xrightarrow{\pi_M} & M \end{array}$$

such that

- $\pi_{\mathcal{G}}: E_{\mathcal{G}} \to \mathcal{G}$ *is a VB of rank* $l+k$ *and* $\pi_M : E_M \to M$ *is a VB of rank* k*;*
- $\mathcal{G} \rightrightarrows M$ *and* $E_{\mathcal{G}} \rightrightarrows E_M$ *are Lie groupoids;*
- *the structure maps* $(\tilde{s}, s), (\tilde{t}, t), (\tilde{m}, m), (\tilde{u}, u), (\tilde{\tau}, \tau)$ *are morphisms of VBs.*

Example 1.2. *Given any Lie groupoid* $\mathcal{G} \rightrightarrows M$*, the tangent manifold of the arrows* $T\mathcal{G} \rightrightarrows TM$ *is a VB-groupoid over* $\mathcal{G} \rightrightarrows M$*.*

Definition-Example 1.3. *Let* $l,k \in \mathbb{N}$ *fixed. On* $\mathrm{GL}(l,k)_0 := \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k) \cong M_{l\times k}(\mathbb{R})$ *we can consider the identity groupoid and two trivial vector bundles,*

$$\mathbb{R}_2^{(l,k)} = \mathbb{R}^l \times \mathbb{R}^k \times \mathrm{GL}(l,k)_0 \quad \textit{and} \quad \mathbb{R}_1^{(l,k)} = \mathbb{R}^k \times \mathrm{GL}(l,k)_0.$$

The space $\mathbb{R}^l$ *acts on* $\mathbb{R}_1^{(l,k)}$ *by* $(c,e,d) \mapsto (dc)+e$ *for* $c \in \mathbb{R}^l$*,* $e \in \mathbb{R}^k$ *and* $d \in \mathrm{GL}(l,k)_0$*. The action groupoid* $\mathbb{R}_2^{(l,k)} \rightrightarrows \mathbb{R}_1^{(l,k)}$ *completes an example of VB-groupoid. We call the VB groupoid* $\mathbb{R}_2^{(l,k)} \rightrightarrows \mathbb{R}_1^{(l,k)}$ *the canonical VB groupoid of rank* (l,k)*.*

Given any VB groupoid $E_{\mathcal{G}} \rightrightarrows E_M$ over a Lie groupoid $\mathcal{G} \rightrightarrows M$ there is a vector bundle over M called the **core**, and given by the set

$$C_M = u^* \ker(\tilde{s}) = \{v \in \ker(\tilde{s})_{u(x)} \subset (E_{\mathcal{G}})_{u(x)} \ \ : \ x \in M\}.$$

The core of $T\mathcal{G}$ as example 1.2 is the Lie algebroid of the Lie groupoid $\mathcal{G}$.

2. VB-groupoids and Poisson geometry

Let us state a list of facts to motivate the relation between VB-groupoids and Poisson geometry. These facts are part of the folklore knowledge but it can be traced to the work "Groupoïdes Symplectiques" of Coste, Dazord and Weinstein in 1987.

- Any Lie groupoid has an associated Lie algebroid. This construction is given by the core of the tangent Lie groupoid. More precisely, the Lie algebroid associated to $\mathcal{G} \rightrightarrows M$ is the core of $T\mathcal{G} \rightrightarrows TM$ given by $C_M = u^* \ker(\tilde{s} = ds)$.
- Any Poisson manifold (M,π) has an associated Lie algebroid $\pi^\sharp\colon T^*M \to TM$.
- For any manifold M the cotangent T^*M is a symplectic manifold and therefore a Poisson manifold. For a Lie groupoid $\mathcal{G} \rightrightarrows M$ the cotangent of $\mathcal{G}$ i.e. $T^*\mathcal{G}$ is a symplectic manifold.
- One of the basic examples of Poisson manifolds is given by the dual of a Lie algebra $\mathfrak{g}$. In this case, linear functions on $\mathfrak{g}^*$ corresponds to elements in $x,y \in \mathfrak{g}$ and for any $\alpha \in \mathfrak{g}^*$ there is $\{x,y\}\alpha = \alpha([x,y])$. There is a similar construction for the dual of a Lie algebroid, so dual of Lie algebroids are also Poisson manifolds. For the Lie algebroid TM we get the symplectic (therefore Poisson) manifold of T^*M.
- The dual of the tangent groupoid $T^*\mathcal{G} \rightrightarrows C_M^*$ is the symplectic groupoid integrating the Poisson manifold C_M^*.
- In particular for a Lie algebra $\mathfrak{g}$ with Lie group G the Lie groupoid $T^*G \rightrightarrows \mathfrak{g}^*$ is symplectic integrating $\mathfrak{g}^*$.

3. Special ordered basis for VB-groupoids

Definition 3.1. *(see [4, section 8]) Let* $E_{\mathcal{G}} \rightrightarrows E_M$ *be a VB-groupoid of rank* (l,k) *over* $\mathcal{G} \rightrightarrows M$*. Its* **fat groupoid** $\tilde{\mathcal{G}}(E_{\mathcal{G}})$ *consists of all the pairs* (g, H_g)*, with* $g \in \mathcal{G}$ *and* $H_g \subseteq (E_{\mathcal{G}})_g$ *any (necessarily* k*-dimensional) subspace such that*

$$H_g \oplus \ker(\tilde{s}_g) = (E_{\mathcal{G}})_g, \qquad H_h \oplus \ker(\tilde{t}_g) = (E_{\mathcal{G}})_g.$$

The set $\tilde{\mathcal{G}}(E_{\mathcal{G}})$ has a natural Lie groupoid structure over M, with structure maps

$$s(g,H_g) = s(g), \qquad t(g,H_g) = t(g), \qquad m\big((g,H_g),(h,H_h)\big) = \big(gh, \tilde{m}(H_g,H_h)\big).$$

$$u(x) = \big(1_x, \tilde{u}(E_M)_x\big), \qquad i(g,H_g) = \big(g^{-1}, \tilde{\tau}(H_g)\big).$$

Moreover, there is an action of $\tilde{\mathcal{G}}(E_{\mathcal{G}} \rightrightarrows M$ on the vector bundle $C_M \times_M E_M \to M$, given by

$$(g,H_g)\cdot(c_x,e_x) := (\ \alpha_g \circ \tilde{R}_{g^{-1}}(c_x) \ \ , \ \tilde{t}_g(\tilde{s}_g|_{H_g})^{-1}(e_x)\ ). \tag{1}$$

for $\alpha_g \in H_g$ such that $\tilde{s}_g(\alpha_g) = \tilde{t}_{1_x}(c_x)$.

Definition 3.2. ***(Special ordered basis,*** *see [1]) Let* $E_{\mathcal{G}} \rightrightarrows E_M$ *be a VB-groupoid of rank* (l,k) *over* $\mathcal{G} \rightrightarrows M$*, we consider the following ordered basis (*OB*)*

$$\mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}} := \big\{(c_g,e_g) \in \mathrm{OB}(E_{\mathcal{G}})_g \mid c_g \in \mathrm{OB}(\ker(\tilde{s})_g); e_g \in \mathrm{OB}(H_g); (g,H_g) \in \tilde{\mathcal{G}}(E_{\mathcal{G}})\big\}\,.$$

This set is called the ***s-bisection frame bundle*** *of* $E_{\mathcal{G}}$*.*

Lemma 3.3. *Let* $E_{\mathcal{G}}$ *be a VB-groupoid. The function*

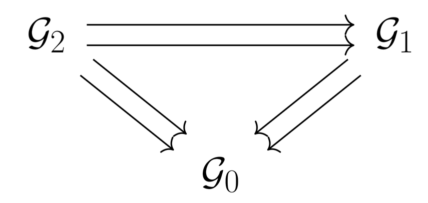
$$F : \mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}) \to \tilde{\mathcal{G}}(E_{\mathcal{G}}) \times_M (\mathrm{Fr}(C_M) \times_M \mathrm{Fr}(E_M))\,;\, (c_g,e_g) \mapsto \big(g, \mathrm{Span}(e_g), \alpha_g^{-}1 \circ (c_g), \tilde{s}(e_g)\big)$$

is a bijection. Therefore $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})$ *has a Lie groupoid structure coming from the action groupoid* $\tilde{\mathcal{G}}(E_{\mathcal{G}}) \times_M (\mathrm{Fr}(C_M) \times_M \mathrm{Fr}(E_M)) \rightrightarrows \mathrm{Fr}(C_M) \times_M \mathrm{Fr}(E_M)$*.*
The map $\tilde{\mathcal{G}}(E_{\mathcal{G}}) \times_M (\mathrm{Fr}(C_M) \times_M \mathrm{Fr}(E_M)) \to \mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}) \hookrightarrow \mathrm{Fr}(E_{\mathcal{G}})$ *is an immersion (i.e.* F *is a diffeomorphism).*

4. The Lie 2 groupoid $\mathrm{GL}(l,k)$ or the matrices acting on $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})$

In general, since we will consider different groupoid structures on the same space, we adopt the notation $s_{ij}, t_{ij}, m_{ij}, u_{ij}, \tau_{ij}$ for the structure maps of a Lie groupoid $\mathcal{G}_i \rightrightarrows \mathcal{G}_j$.

Definition 4.1. *(see [3]) A (strict)* **Lie 2-groupoid** $\mathcal{G}_2 \rightrightarrows \mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ *is a double Lie groupoid where the base groupoid* $\mathcal{G}_0 \rightrightarrows M = \mathcal{G}_0$ *is the unit groupoid. In other words it is a commutative diagram of Lie groupoids*



such that the following two conditions are satisfied:

1. all the source and targets maps are Lie groupoid morphisms;

2. the interchange law

$$(g_1 \circ_{m_{20}} g_2) \circ_{m_{21}} (g_3 \circ_{m_{20}} g_4) = (g_1 \circ_{m_{21}} g_3) \circ_{m_{20}} (g_2 \circ_{m_{21}} g_4)$$

holds for all $g_i \in \mathcal{G}_2$ *such that the compositions above make sense;*

We will focus in a particular Lie 2-groupoid which we describe here below:

Definition-Example 4.2. *For any pair* (l,k) *of natural numbers the* ***general linear 2-groupoid*** *of rank* (l,k)*, denoted by* $\mathrm{GL}(l,k)$*, is the Lie 2-groupoid with*

$$\mathrm{GL}(l,k)_2 := \left\{ \left(d, \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix}\right) \in \mathrm{GL}(l,k)_0 \times \mathrm{GL}(l+k) \ : \ (I_l + Jd) \in \mathrm{GL}(l) \text{ and } (I_k + dJ) \in \mathrm{GL}(k) \right\}$$

$$\mathrm{GL}(l,k)_1 := \mathrm{GL}(l,k)_0 \times \mathrm{GL}(l) \times \mathrm{GL}(k),$$
$$\mathrm{GL}(l,k)_0 := \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k) \cong M_{l\times k}(\mathbb{R}).$$

- *The groupoid structure on* $\mathrm{GL}(l,k)_2 \rightrightarrows \mathrm{GL}(l,k)_0$ *is like a right action groupoid with:*

$$d \xleftarrow{t_{20}} \left(d, \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix}\right) \xmapsto{s_{20}} ((I+dJ)B)^{-1}dA \ .$$

- *The groupoid structure on* $\mathrm{GL}(l,k)_2 \rightrightarrows \mathrm{GL}(l,k)_1$ *is the unique one with:*

$$(d\,,\, A\,,\, (I+dJ)B) \xleftarrow{t_{21}} \left(d\,, \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix}\right) \xmapsto{s_{21}} (d\,,\, (I+Jd)^{-1}A\,,\, B) \ ,$$

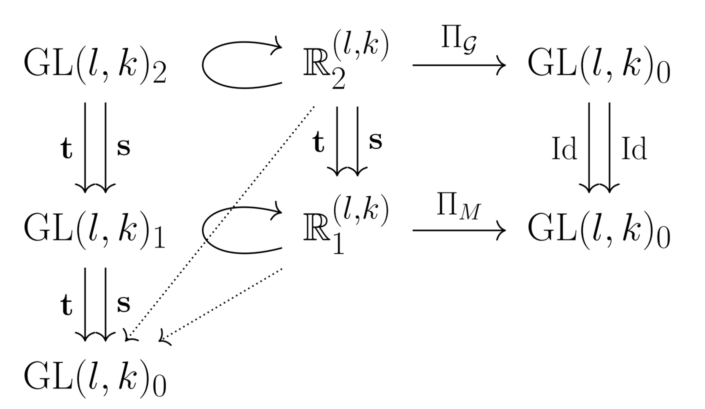
and composition given by:

$$\left(\left(d, \begin{pmatrix} A & J(I+dJ')B' \\ 0 & (I+dJ')B' \end{pmatrix}\right), \left(d, \begin{pmatrix} (I+Jd)^{-1}A & J'B' \\ 0 & B' \end{pmatrix}\right) \right) \xmapsto{m_{21}} \left(d, \begin{pmatrix} A & (JdJ'+J+J')B' \\ 0 & B' \end{pmatrix}\right)$$

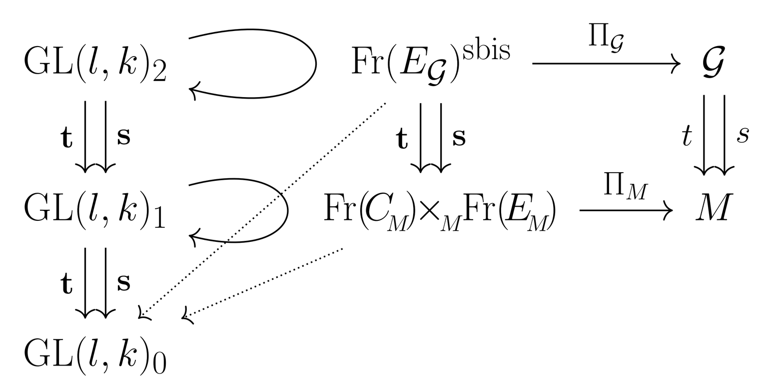
- *The groupoid* $\mathrm{GL}(l,k)_1 \rightrightarrows \mathrm{GL}(l,k)_0$ *is a right groupoid action with:*

$$d \xleftarrow{t_{10}} (d,A,B) \xmapsto{s_{10}} B^{-1}dA \ .$$

Proposition 4.3. *(see [1]) The usual left action of* $\mathrm{GL}(l+k)$ *on* $\mathbb{R}^l \times \mathbb{R}^k$ *induces a* $\mathrm{GL}(l,k)_1 \rightrightarrows \mathrm{GL}(l,k)_0$ *action on* $\mathbb{R}_2^{(l,k)}$ *as in 1.3, and a canonical* $\mathcal{G}(l,k)_1 \rightrightarrows \mathrm{GL}(l,k)_0$ *action on* $\mathbb{R}_0^{(l,k)}$ *extending to a Lie 2-groupoid (left) action:*



Proposition 4.4. *(see [1]) The usual right action of* $\mathrm{GL}(l+k)$ *on* $\mathrm{Fr}(E_{\mathcal{G}})$ *induces a principal* $\mathrm{GL}(l,k)_1 \rightrightarrows \mathrm{GL}(l,k)_0$ *action on* $\mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}}$*, and a canonical* $\mathcal{G}(l,k)_1 \rightrightarrows \mathrm{GL}(l,k)_0$ *action on* $\mathrm{Fr}(C_M) \times_M \mathrm{Fr}(E_M)$ *extending to a Lie 2-groupoid action:*



We called PB-groupoid a diagram as above, with a principal Lie 2-groupoid action on a Lie groupoid. Using the same construction of a VB from a PB we proved that the following correspondence is true:

$$\big\{\ \mathrm{GL}(l,k)\text{-PB-groupoids over } \mathcal{G} \rightrightarrows M \big\} \longleftrightarrow \big\{\ \text{VB-groupoids of rank } (l,k) \text{ over } \mathcal{G} \rightrightarrows M \big\}$$

References

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