Quantization (Deformation)

Introducction

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Quantization

Classical M $\longleftrightarrow$ Quantum M

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Quantization

1

Classical M $\longleftrightarrow$ Quantum M

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Quantization

1

Classical M $\longleftrightarrow$ Quantum M

2

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3

Quantization

1

2

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Quantization

(Deformation) 4

1 Classical M $\longleftrightarrow$ 2 Quantum M

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1 Classical Mechanics

The Model

- A Poisson manifold $(M, \{-,-\})$
- A Hamiltonian $H \in C^\infty(M)$

⇒ The evolution equation is

$$\frac{d}{dt} f = \{H, f\}$$

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  - A Poisson manifold $(M, \{-,-\})$
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- **The main example**
  - $M = \mathbb{R}^{2n}$ coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$

    $$\{f, g\} = \sum_i \left(\frac{\partial}{\partial p_i} f\right)\left(\frac{\partial}{\partial q_i} g\right) - \left(\frac{\partial}{\partial p_i} g\right)\left(\frac{\partial}{\partial q_i} f\right)$$

  - $H(p, q) = \frac{\|p\|^2}{2} + V(q)$

$\Rightarrow$

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$$\{f, g\} = \sum_i \left(\frac{d}{dp_i} f\right)\left(\frac{d}{dq_i} g\right) - \left(\frac{d}{dp_i} g\right)\left(\frac{d}{dq_i} f\right)$$

  - $H(p, q) = \frac{\|p\|^2}{2} + V(q)$

⇒ The evolution equation is equivalent to

$$\left.\begin{array}{l} \boxed{\dfrac{d p_i}{dt} = \{H, p_i\} = -\dfrac{dH}{dq_i}} \\[2ex] \boxed{\dfrac{d q_i}{dt} = \{H, q_i\} = \dfrac{dH}{dp_i}} \end{array}\right\} \begin{array}{c} \text{Hamilton-Jacobi} \\ \text{equations} \\ \text{(Euler-Lagrange)} \end{array}$$

if these equations aren't familiar,

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if these equations aren't familiar, for $H = \frac{\|p\|^2}{2} + V(q)$

$$\Rightarrow \frac{\partial p_i}{\partial t} = -\partial_{q_i} V(q)$$

$$\frac{\partial q_i}{\partial t} = p_i$$

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if these equations aren't familiar, for $H = \frac{\|p\|^2}{2} + V(q)$

$$\Rightarrow \left.\begin{array}{l} \frac{\partial p_i}{\partial t} = -\partial_{q_i} V(q) \\ \frac{\partial q_i}{\partial t} = p_i \end{array}\right\} \quad \text{Newton's Eq in } \mathbb{R}^n \quad \boxed{\underbrace{\ddot{q}_i}_{\text{Acceleration}} = -\underbrace{\partial_{q_i} V(q)}_{\text{Force}}}$$

Quantum Mechanics

- ## The Model

  - A Hilbert space L (states) &
    a subalgebra $A \subseteq B(L)$ (observables)
  - Srödinger operator $\hat{H}$ on L

$\Rightarrow$ The evolution equation is

$$\underbrace{i\hbar \frac{\partial f}{\partial t} = \hat{H}(f)}_{\text{For states}} \qquad \underbrace{i\hbar \frac{d}{dt} Q = [\hat{H}, Q]}_{\text{for observables}}$$

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The main example

- $L \subset L^2(\mathbb{R}^n)$ & $A \subseteq \mathcal{B}(L)$ (Let it be vague)
- $\hat{H} = -\frac{\hbar^2}{2}\Delta + V \longleftarrow V \in C^\infty(M)$

$\Delta = \partial_{q_1}^2 + \dots + \partial_{q_n}^2$ the laplacian

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⇒ Evolution equation is

$$\left[\begin{array}{l} \underline{\text{states}} \quad f \in L \\ i\hbar\, \partial_t f = -\frac{\hbar^2}{2} \Delta(f) + V \cdot f \end{array}\right.$$

Srödinger's equation

* For observables Note that

1 Quantum Mechanics

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    ↳ $\partial^2_{q_1} + \dots + \partial^2_{q_n}$ the laplacian

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* For observables Note that

$$d_t(Qf) = (\partial_t Q) f + Q(\partial_t f) \leftarrow \text{Leibnitz}$$

Quantum Mechanics

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  - A Hilbert space L (states) &
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  - Schrödinger operator $\hat{H}$ on L

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  - $L \subset L^2(\mathbb{R}^n)$ & $A \subseteq B(L)$ (Let it be vague)
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$$\boxed{\begin{aligned}&\text{states} \quad f \in L \\ &i\hbar \partial_t f = -\frac{\hbar^2}{2}\Delta(f) + V \cdot f\end{aligned}}$$

← Schrödinger's equation

* For observables Note that

$$d_t(Qf) = (\partial_t Q) f + Q(\partial_t f) \quad \leftarrow \text{Leibnitz}$$

$$\frac{1}{i\hbar}\hat{H}(Qf) = (\partial_t Q) f + \frac{1}{i\hbar} Q(\hat{H} f)$$

$$\Rightarrow i\hbar(\partial_t Q)(f) = [\hat{H}, Q](f) \quad \text{eq for observables}$$

3 Example: Quantization example: Harmonic Oscilator

Quantum

for $f \in C^\infty(\mathbb{R} \times \mathbb{R})$

$$\underbrace{i\hbar \frac{d}{dt} f = -\frac{\hbar^2}{2} \frac{d^2}{dq^2} f - \frac{q^2 f}{2}}_{\text{Srödinger's}}$$

Classic

$$\underbrace{\frac{d^2}{dt^2} q = q^2}_{\text{Newton's}} \quad \text{for } q \in C^\infty(\mathbb{R})$$

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$$\left.\begin{array}{l} \text{let } p := \frac{d}{dt} q \\ \Rightarrow \frac{dP}{dt} = q^2 \end{array}\right\} \text{Hamilton-Jacoby eq for}$$

$$H(p,q) = \frac{P^2}{2} - \frac{q^2}{2} \quad \text{in } M = \mathbb{R}^2$$

- This system have the symmetries

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- This system have the symetries

$a(p,q) = P + q$ & $a^\dagger(p,q) = -P + q$

$\Rightarrow$

$\{H, a\} = P + q = a$

$\{H, a^\dagger\} = P - q = -a^\dagger$

3 Example: Quantization example: Harmonic Oscilator

Quantum

$$\underbrace{i\hbar \frac{d}{dt} f = -\frac{\hbar^2}{2}\frac{d^2 f}{dq^2} - \frac{q^2 f}{2}}_{\text{Srödinger's}}$$

for $f \in C^\infty(\mathbb{R}\times\mathbb{R})$

Symmetries:

$\Rightarrow$ $\hat{a} = i\hbar \frac{d}{dq} + q$ | $\hat{a}^+ = -i\hbar \frac{d}{dq} + q$

$[\hat{H}, \hat{a}] = i\hbar\, \hat{a}$ | $[\hat{H}, \hat{a}^+] = -i\hbar\, \hat{a}^+$

Let f_0 an eigenvector for $\hat{H}$ $\Rightarrow$ $H(f_0) = \lambda f_0$

Classic

$$\underbrace{\frac{d^2}{dt^2} q = q^2}_{\text{Newton's}} \quad \text{for } q \in C^\infty(\mathbb{R})$$

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$[\hat{H}, \hat{a}] = i\hbar\, \hat{a}$ | $[\hat{H}, \hat{a}^+] = -i\hbar\, \hat{a}^+$

Let f_0 an eigenvector for $\hat{H}$ $\Rightarrow$ $\hat{H}(f_0) = \lambda f_0$

$\Rightarrow$ $\hat{H}(\hat{a} f_0) = (\lambda + i\hbar)(\hat{a} f_0)$

$\hat{H}(\hat{a} f_0) = (\lambda - i\hbar)(\hat{a}^+ f_0)$

so $\hat{a} f_0$ & $\hat{a}^+ f_0$ are eigen-vectors

$\hat{a}$ creation $\quad$ $\hat{a}^+$ destruction

Hermite Polinomials

Classic

$$\underbrace{\frac{d^2}{dt^2} q = q^2 \text{ for } q \in C^\infty(\mathbb{R})}_{\text{Newton's}}$$

Let $p := \frac{d}{dt} q$

$\Rightarrow \frac{dP}{dt} = q^2$

Hamilton-Jacoby eq for $H(p,q) = \frac{P^2}{2} - \frac{q^2}{2}$ in $M = \mathbb{R}^2$

- This system have the symetries

$a_{(p,q)} = P + q$ & $a^+_{(p,q)} = -P + q$

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$\{H, a\} = P + q = a$

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3 Quantization Dream

$$\begin{array}{ccccc} \text{Poisson Mfd } M & \longleftarrow & \boxed{\begin{matrix}Q\\U\\A\\N\\T\end{matrix}} & \longrightarrow & \text{Hilbert space } L_M \\ \text{any function } f \in C^\infty(M) & \longleftarrow & & \longrightarrow & \text{operator } \hat{f} : L_M \to L_M \end{array}$$

such that:

1. $f \to \hat{f}$ is $\mathbb{C}$-linear
2. $[\hat{f}, \hat{g}] = i\hbar \widehat{\{f, g\}}$
3. $\widehat{\dfrac{\|p\|^2}{2} + V(q)} = -\dfrac{\hbar^2}{2}\Delta + V$
4. for P a power series in $\mathbb{R}$

$$\widehat{P(f)} = P(\hat{f})$$

3 Quantization Dream

Poisson Mfd M $\longleftarrow$ $\boxed{\text{QUANT}}$ $\longrightarrow$ Hilbert space L_M

any function $f \in C^\infty(M)$ $\longleftarrow$ $\longrightarrow$ operator $\hat{f} : L_M \to L_M$

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Why we want this?

1. convenience
2. if $\partial_t f = \{H, f\}$

$\Rightarrow \partial_t \hat{f} = \dfrac{1}{i\hbar}[\hat{H}, \hat{f}]$

solutions to solutions
symetries to symetries

3 Quantization Dream

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   $\Big\}$ solutions to solutions, symetries to symetries
3. It fits in the known examples
4. It is very convenient. For example to solve
   $$\partial_t f = \hat{g}\, f$$
   the natural solution is: $f = e^{t\hat{g}} f_0$ if exists
   and it would because $e^{t\hat{g}} = \widehat{e^{tg}}$

3 Quantization Dream ☆

~~Poisson Mfd M $\longleftarrow$ QUANT $\longrightarrow$ Hilbert space L_M~~

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~~$$\widehat{P(f)} = P(\hat{f})$$~~

Why we want this?

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2. if $\partial_t f = \{H, f\}$ $\Rightarrow$ $\partial_t \hat{f} = \frac{1}{i\hbar}[\hat{H}, \hat{f}]$ } solutions to solutions, symetries to symetries
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☆ PROBLEM! This Dream is a Lie!

3 Real Quantization (still using $\mathbb{C}$)

Poisson Mfd M $\longleftarrow$ QUANT $\longrightarrow$ A Hilbert space L_M

$\forall f \in A \subseteq C^\infty(M)$ $\longleftarrow$ QUANT $\longrightarrow$ an operator

A ← subalgebra

$$\hat{f} : \mathrm{Dom}(\hat{f}) \longrightarrow L_M, \quad \mathrm{Dom}(\hat{f}) \subseteq L_M$$

such that

1. $f \mapsto \hat{f}$ is $\mathbb{C}$-linear
2. $[\hat{f}, \hat{g}] = i\hbar \widehat{\{f,g\}} + \mathcal{O}(\hbar^2)$
3. $\widehat{\left(\frac{\|P\|^2}{2} + V(q)\right)} = -\frac{\hbar^2}{2}\Delta + V$
4. for P a power series in $\mathbb{R}$

$$P(\hat{f}) = \widehat{P(f)} + \mathcal{O}(\hbar)$$

3 Real Quantization (still using $\mathbb{C}$)

$$\text{Poisson Mfd } M \longleftarrow \boxed{\text{QUANT}} \longrightarrow \text{A Hilbert space } L_M$$

$$\forall f \in \underset{\substack{\uparrow \\ \text{subalgebra}}}{A} \subseteq C^\infty(M) \longleftarrow \quad \longrightarrow \text{an operator } \hat{f} : \underset{\substack{\cap \\ L_M}}{\mathrm{Dom}(\hat{f})} \longrightarrow L_M$$

such that

1. $f \longmapsto \hat{f}$ is $\mathbb{C}$-linear
2. $[\hat{f}, \hat{g}] = i\hbar \widehat{\{f, g\}} + O(\hbar^2)$
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4. for P a power series in $\mathbb{R}$

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Example $M = \mathbb{R}^{2n}$

in $\mathbb{R}^n$ there is the Fourier transform for any $f \in C_c^\infty(\mathbb{R}^n)$ it is

$$\boxed{\mathcal{F}(f)(p) = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}\langle p, q' \rangle} f(q')\, dq'}$$

- it satisfy:

3 Real Quantization (still using $\mathbb{C}$)

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- it satisfy: $\mathcal{F}(\partial_{q_i} f)(p) = \frac{-i}{\hbar} p_i \, \mathcal{F}(f)(p)$
- Therefore: $\mathcal{F}\left(-\frac{\hbar^2}{2}\Delta(f)\right) = \frac{\|p\|^2}{2}\mathcal{F}(f)(p)$

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Example $M = \mathbb{R}^{2n}$

in $\mathbb{R}^n$ there is the Fourier transform

for any $f \in C_c^\infty(\mathbb{R}^n)$ it is

$$\boxed{\mathcal{F}(f)(p) = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}\langle p, q' \rangle} f(q')\, dq'}$$

- it satisfy: $\mathcal{F}(\partial_{q_i} f)(p) = \frac{-i}{\hbar} p_i \, \mathcal{F}(f)(p)$
- Therefore: $\mathcal{F}\left(-\frac{\hbar^2}{2}\Delta(f)\right) = \frac{\|p\|^2}{2}\mathcal{F}(f)(p)$
- this help us rewrite the Srödinger operator as:

$$\hat{H}(f)(q) = \mathcal{F}^{-1}\left(\mathcal{F}\left(-\frac{\hbar^2}{2}\Delta(f) + Vf\right)\right) q$$

$$= \iint e^{\mathrm{Bla}(q-q')} \underbrace{H(p, q')} f(q')\, dq'\, dP$$

3 Real Quantization (still using $\mathbb{C}$)

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$\forall f \in A \subseteq C^\infty(M)$ ⟵ QUANT ⟶ an operator

subalgebra

$$\hat{f} : \mathrm{Dom}(\hat{f}) \longrightarrow L_M, \quad \mathrm{Dom}(\hat{f}) \subseteq L_M$$

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Example $M = \mathbb{R}^{2n}$

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Idea

Poisson $M \xrightarrow{DQ} C^*$-Algebra A_M

$f \in C^\infty(M) \mapsto a_f \in A_M$

Not DQ

Gelfand-Neimark-Segal

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Quantization[3]

(Deformation)[4]

Classical M [1] $\longleftrightarrow$ Quantum M [2]

Alfonso Garmendia

CRM

Xq!

1 Classical Mechanics

- The Model
  - A Poisson manifold $(M, \{-,-\})$
  - A Hamiltonian $H \in C^\infty(M)$
- $\Rightarrow$ The evolution equation is

$$\frac{d}{dt} f = \{H, f\}$$

2 Quantum Mechanics

- The Model
  - A Hilbert space L (states) &
    a subalgebra $A \subseteq B(L)$ (observables)
  - Srödinger operator $\hat{H}$ on L
- $\Rightarrow$ The evolution equation is

$$\underbrace{i\hbar \frac{\partial}{\partial t} f = \hat{H}(f)}_{\text{For states}} \qquad \underbrace{i\hbar \frac{d}{dt} Q = [\hat{H}, Q]}_{\text{for observables}}$$

3 Real Quantization (still using $\mathbb{C}$)

Poisson Mfd M $\longleftarrow$ QUANT $\longrightarrow$ A Hilbert space L_M

$\forall f \in A \subseteq C^\infty(M)$ (subalgebra) $\longleftarrow$ QUANT $\longrightarrow$ an operator $\hat{f} : Dom(\hat{f}) \to L_M$, $Dom(\hat{f}) \subseteq L_M$

such that

1. $f \mapsto \hat{f}$ is $\mathbb{C}$-linear
2. $[\hat{f}, \hat{g}] = i\hbar \widehat{\{f,g\}} + \mathcal{O}(\hbar^2)$
3. $\widehat{\frac{\|P\|^2}{2} + V(q)} = -\frac{\hbar^2}{2}\Delta + V$
4. for P a power series in $\mathbb{R}$
   $P(\hat{f}) = \widehat{P(f)} + \mathcal{O}(\hbar)$

Example $M = \mathbb{R}^{2n}$

in $\mathbb{R}^n$ there is the Fourier transform for any $f \in C_c^\infty(\mathbb{R}^n)$ it is

$$\boxed{\mathcal{F}(f)_{(P)} = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}\langle P, q'\rangle} f(q')\, dq'}$$

- it satisfy: $\mathcal{F}(\partial_{q_i} f)(P) = \frac{i}{\hbar} P_i \, \mathcal{F}(f)(P)$
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3 Example: Quantization example: Harmonic Oscilator

Quantum

$$\underbrace{i\hbar \frac{d}{dt} f = -\frac{\hbar^2}{2}\frac{d^2 f}{dq^2} - \frac{q^2 f}{2}}_{\text{Srödinger's}}$$

for $f \in C^\infty(\mathbb{R} \times \mathbb{R})$

Symetries:

$\Rightarrow$ $\hat{a} = i\hbar \frac{d}{dq} + q$ $\qquad$ $\hat{a}^+ = -i\hbar \frac{d}{dq} + q$

$[\hat{H}, \hat{a}] = i\hbar\, \hat{a}$ $\qquad$ $[\hat{H}, \hat{a}^+] = -i\hbar\, \hat{a}^+$

Let f_0 an eigenvector for $\hat{H}$ $\Rightarrow$ $\hat{H}(f_0) = \lambda f_0$

$\Rightarrow$ $\hat{H}(\hat{a} f_0) = (\lambda + i\hbar)(\hat{a} f_0)$

$\hat{H}(\hat{a} f_0) = (\lambda - i\hbar)(\hat{a}^+ f_0)$

so $\hat{a} f_0$ & $\hat{a}^+ f_0$ are eigen-vectors

$\hat{a}$ creation $\qquad$ $\hat{a}^+$ destruction

Hermite Polinomials

Classic

$$\underbrace{\frac{d^2}{dt^2} q = q^2 \quad \text{for } q \in C^\infty(\mathbb{R})}_{\text{Newton's}}$$

let $P := \frac{d}{dt} q$

$\Rightarrow$ $\frac{dP}{dt} = q^2$

Hamilton-Jacoby eq for

$$H(p,q) = \frac{P^2}{2} - \frac{q^2}{2} \quad \text{in } M = \mathbb{R}^2$$

- This system have the symetries

$a(p,q) = P + q$ & $a^+(p,q) = -P + q$

$\Rightarrow$

$\{H, a\} = P + q = a$

$\{H, a^+\} = P - q = -a^+$

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