

Quantization (Deformation)

Introduction

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Quantization

Classical M $\longleftrightarrow$ Quantum M

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Quantization

1

Classical M



Quantum M

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Quantization

1

Classical M



2

Quantum M

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3

Quantization

1

Classical M



2

Quantum M

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3

Quantization

(Deformation)⁴

1

Classical M



2

Quantum M

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1 Classical Mechanics

• The Model

• A Poisson manifold $(M, \{-, -\})$

• A Hamiltonian $H \in C^\infty(M)$

⇒ The evolution equation is

$$\frac{d}{dt} f = \{H, f\}$$

□

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• The main example

- $M = \mathbb{R}^{2n}$ coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial p_i} \right) \left(\frac{\partial g}{\partial q_i} \right) - \left(\frac{\partial g}{\partial p_i} \right) \left(\frac{\partial f}{\partial q_i} \right)$$

- $H(p, q) = \frac{\|p\|^2}{2} + V(q)$

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equations

(Euler-Lagrange)

if these equations aren't familiar,

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$$\Rightarrow \frac{dp_i}{dt} = -\partial_{q_i} V(q)$$

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Newton's Eq in $\mathbb{R}^n$

$$\boxed{\ddot{q}_i = -\partial_{q_i} V(q)}$$

Acceleration

Force



Quantum Mechanics

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- A Hilbert space L ^{states} &
a subalgebra $A \subseteq B(L)$ ^{observables}
- Schrödinger operator $\hat{H}$ on L

⇒ The evolution equation is

$$\underbrace{i\hbar \frac{\partial}{\partial t} f = \hat{H}(f)}_{\text{For states}}$$

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- $L \subset L^2(\mathbb{R}^n)$ & $A \subseteq B(L)$ (let it be vague)

$$\hat{H} = -\frac{\hbar^2}{2} \Delta + V \leftarrow V \in C^\infty(M)$$

↳ $\partial_{q_1}^2 + \dots + \partial_{q_n}^2$ the Laplacian

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$$d_t(Qf) = (\partial_t Q)f + Q(\partial_t f) \leftarrow Leibniz$$

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$$\partial_t (Qf) = (\partial_t Q)f + Q(\partial_t f) \quad \leftarrow \text{Leibniz}$$

$$\stackrel{||}{=} \frac{1}{i\hbar} \hat{H}(Qf) = (\partial_t Q)f + \frac{1}{i\hbar} Q(\hat{H}f)$$

$$\Rightarrow i\hbar (\partial_t Q)(f) = [\hat{H}, Q](f) \quad \text{eq for observables}$$



3 Example: Quantization example: Harmonic Oscillator

Quantum

for $f \in C^\infty(\mathbb{R} \times \mathbb{R})$

$$\underbrace{i\hbar \frac{d}{dt} f = -\frac{\hbar^2}{2} \frac{d^2}{dq^2} f - \frac{q^2}{2} f}_{\text{Schrödinger's}}$$

Classic

$$\underbrace{\frac{d^2}{dt^2} q = -q}_{\text{Newton's}} \text{ for } q \in C^\infty(\mathbb{R})$$

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$$\left. \begin{array}{l} \text{Hamilton-Jacobi eq for} \\ H(p, q) = \frac{p^2}{2} - \frac{q^2}{2} \text{ in } M = \mathbb{R}^2 \end{array} \right\}$$

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• This system have the symmetries

$$a(p, q) = p + q \quad \& \quad a^+(p, q) = -p + q$$

$\Rightarrow$

$$\{H, a\} = p + q = a$$

$$\{H, a^+\} = -p + q = -a^+$$



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Symmetries:

$$\Rightarrow \hat{a} = i\hbar \frac{d}{dq} + q \quad \left| \quad \hat{a}^\dagger = -i\hbar \frac{d}{dq} + q\right.$$
$$[\hat{H}, \hat{a}] = i\hbar \hat{a} \quad \left| \quad [\hat{H}, \hat{a}^\dagger] = -i\hbar \hat{a}^\dagger\right.$$

Let f_0 an eigenvector for $\hat{H} \Rightarrow H(f_0) = \lambda f_0$

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$$\Rightarrow \hat{H}(\hat{a} f_0) = (\lambda + i\hbar)(\hat{a} f_0) \quad \hat{a} f_0 \text{ \& \& } \hat{a}^\dagger f_0$$

$$\hat{H}(\hat{a}^\dagger f_0) = (\lambda - i\hbar)(\hat{a}^\dagger f_0) \quad \text{are eigen-vectors}$$

$\hat{a}$ creation $\hat{a}^\dagger$ destruction 

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3 Quantization Dream

Poisson Mfd $M \leftarrow \begin{matrix} Q \\ U \\ A \\ N \\ T \end{matrix} \rightarrow$ Hilbert space L_M
any function $f \in C^\infty(M) \leftarrow \begin{matrix} Q \\ U \\ A \\ N \\ T \end{matrix} \rightarrow$ operator $\hat{f} : L_M \rightarrow L_M$

such that:

(1) $f \rightarrow \hat{f}$ is $\mathbb{C}$ -linear

(2) $[\hat{f}, \hat{g}] = i\hbar \widehat{\{f, g\}}$

(3) $\widehat{\frac{\|p\|^2}{2} + V(q)} = -\frac{\hbar^2}{2} \Delta + V$

(4) for P a power series in $\mathbb{R}$

$$\widehat{P(f)} = P(\hat{f})$$

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Why we want this? (1) convenience

(2) if $\partial_t f = \{H, f\}$
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(4) It is very convenient. For example to solve

$$\partial_t f = \hat{g} f$$

the natural solution is: $f = e^{t\hat{g}} f_0$ if exists
 and it would because $e^{t\hat{g}} = \widehat{e^{tg}}$

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★

PROBLEM! This Dream is a lie!

3 Real Quantization (still using $\mathbb{C}$)

$$\begin{array}{ccc} \text{Poisson Mfd } M & \leftarrow \begin{array}{|c|} \hline Q \\ \hline V \\ \hline \Delta \\ \hline N \\ \hline T \\ \hline \end{array} & \rightarrow \text{A Hilbert space } L_M \\ \forall f \in A \subseteq C^\infty(M) & \leftarrow & \rightarrow \text{an operator} \\ \uparrow & & \hat{f} : \text{Dom}(\hat{f}) \rightarrow L_M \\ \text{subalgebra} & & \cap \\ & & L_M \end{array}$$

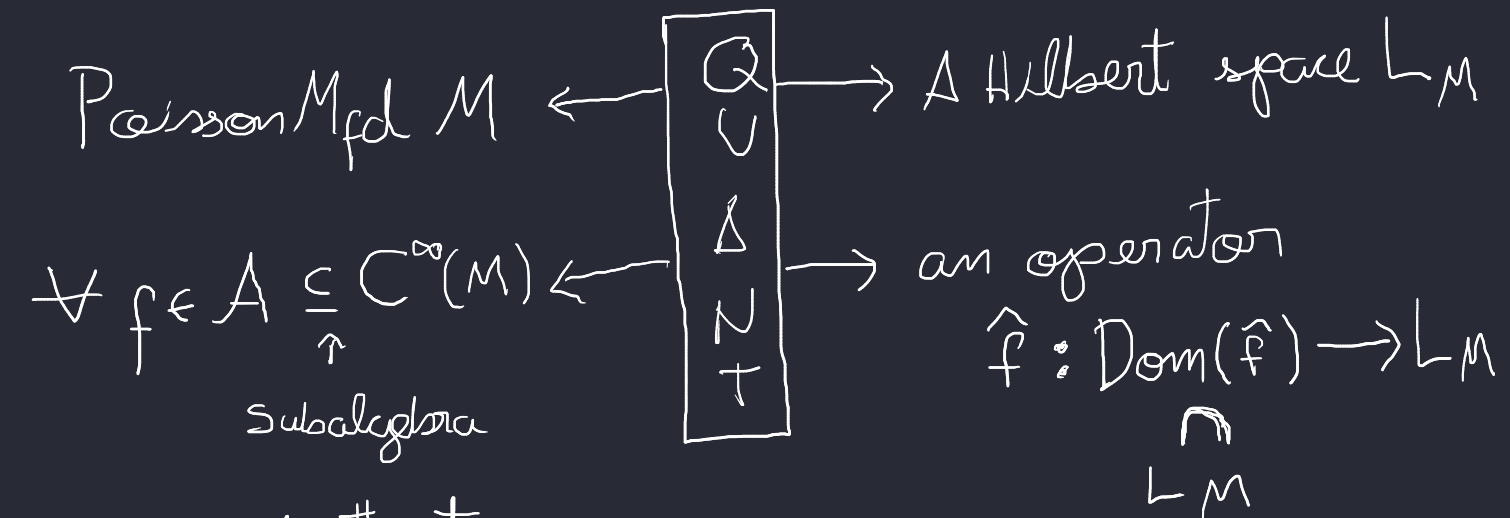
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$$P(\hat{f}) = \widehat{P(f)} + \mathcal{O}(\hbar)$$



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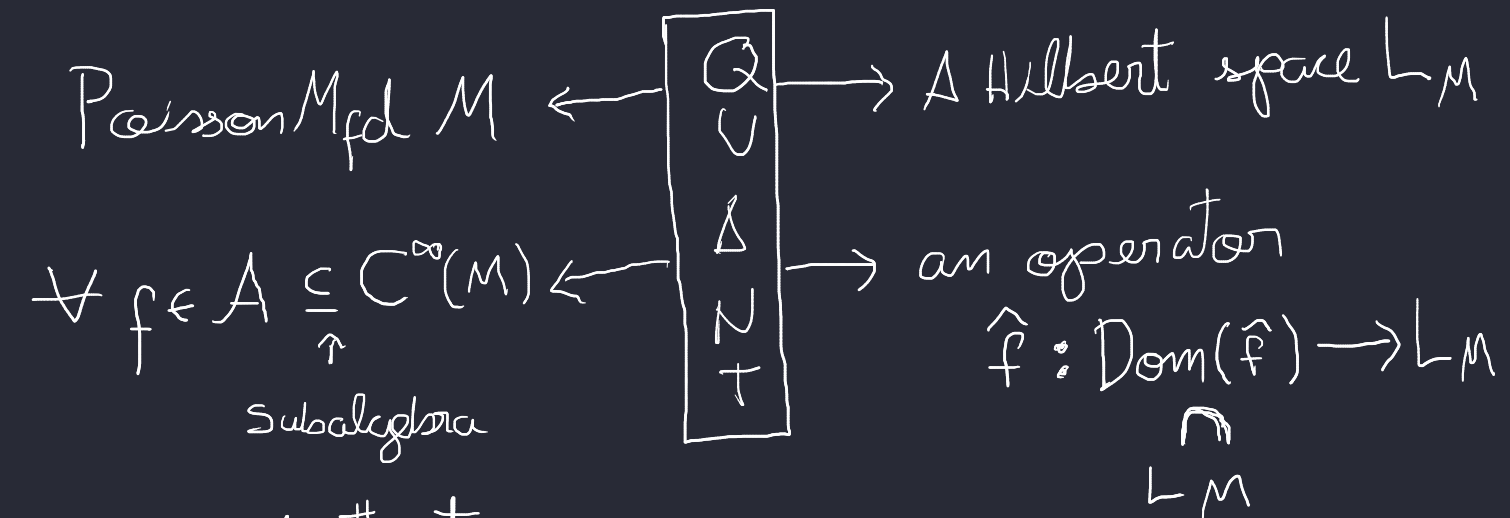
Example $M = \mathbb{R}^{2n}$

in $\mathbb{R}^n$ there is the Fourier transform
 for any $f \in C_c^\infty(\mathbb{R}^n)$ it is

$$\mathcal{F}(f)(p) = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} \langle p, q' \rangle} f(q') dq'$$

- it satisfy:

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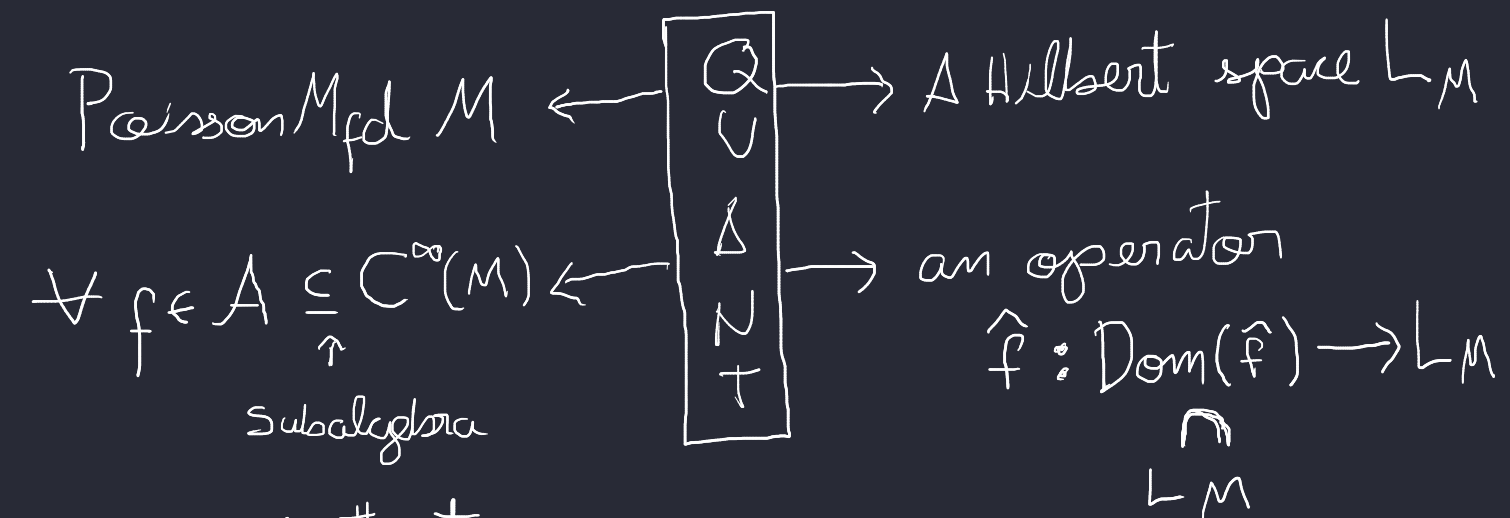
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- Therefore: $\mathcal{F}\left(-\frac{\hbar^2}{2} \Delta(f)\right) = \frac{\|p\|^2}{2} \mathcal{F}(f)(p)$

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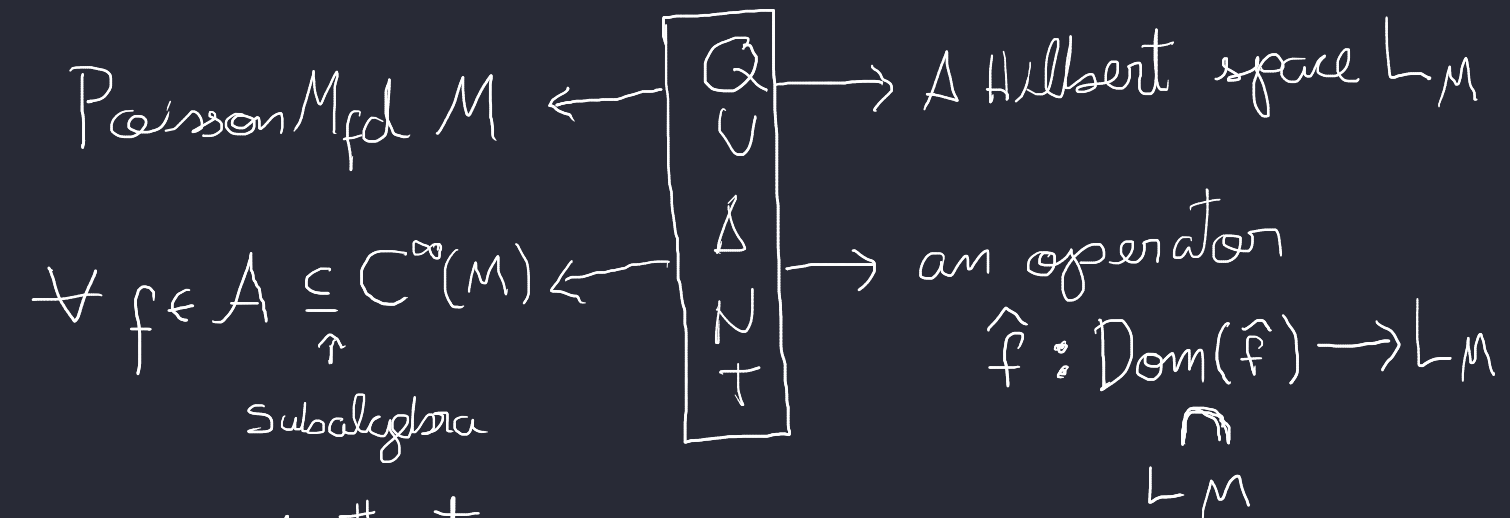
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- This help us rewrite the Schrödinger operator as:

$$\hat{H}(f)(q) = \mathcal{F}^{-1}\left(\mathcal{F}\left(-\frac{\hbar^2}{2} \Delta(f) + V f\right)\right)_q$$

$$= \iint e^{i \text{Bl}_0(q, q')} \underbrace{H(p, q')}_{\text{orange box}} f(q') dq' dp$$

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$$\hat{H}(f)(q) = \mathcal{F}^{-1}\left(\mathcal{F}\left(-\frac{\hbar^2}{2} \Delta(f) + V f\right)\right)(q)$$

$$= \iint e^{i \text{Bl}_a(q-q')} \underbrace{H(P, q')}_{\text{}} f(q') dq' dP$$

4 Formal Deformation Quantization

Idea

Poisson $M \xrightarrow{\text{DQ}} C^*\text{-Algebra } A_M$

$$f \in C^\infty(M) \mapsto a_f \in A_M$$

Not DQ

Gelfand-Neimark-Segal
Rep theory

Hilbert L_M

$$\longmapsto \hat{f}_M: \text{Dom}(\hat{f}) \rightarrow L_M$$

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How

- For $M \xrightarrow{DQ} (C^\infty(M)[[\hbar]], *)$

Power series | Not usual product

- $f \in C^\infty(M) \mapsto \hat{f} \equiv f + O(\hbar) + O(\hbar^2) + \dots$

- $f * g = f \cdot g + \hbar B_1(f, g) + \hbar^2 B_2(f, g) + \dots \quad \textcircled{1}$

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How

- For $M \xrightarrow{DQ} (C^\infty(M)[[\hbar]], *)$
Power series | Not usual product
- $f \in C^\infty(M) \mapsto \hat{f} = f + O(\hbar) + O(\hbar^2) + \dots$
- $f * g = f \cdot g + \hbar B_1(f, g) + \hbar^2 B_2(f, g) + \dots \quad \textcircled{1}$

such that:

- The product $*$ is associative (otherwise not a $C^*-Algebra$)
- $[a_f, a_g] = f * g - g * f = i\hbar \{f, g\} + O(\hbar^2)$

This gives some restrictions

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How

- For $M \xrightarrow{\text{DQ}} (C^\infty(M)[[\hbar]], *)$
Power series | Not usual product
- $f \in C^\infty(M) \mapsto \hat{f} = f + 0\hbar + 0\hbar^2 + \dots$
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- There exist a product $*$ for any Poisson M
 $*$ is unique up to something non trivial

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How

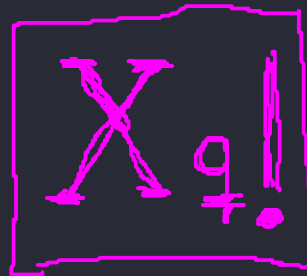
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1 Classical Mechanics

The Model

- A Poisson manifold $(M, \{-, -\})$
- A Hamiltonian $H \in C^\infty(M)$

⇒ The evolution equation is

$$\frac{d}{dt} f = \{H, f\}$$

2 Quantum Mechanics

The Model

- A Hilbert space L & a subalgebra $A \subseteq B(L)$ ^{states} _{observables}
- Schrödinger operator $\hat{H}$ on L

⇒ The evolution equation is

$$\underbrace{i\hbar \frac{d}{dt} f = \hat{H}(f)}_{\text{for states}} \quad \underbrace{i\hbar \frac{d}{dt} A = [\hat{H}, A]}_{\text{for observables}}$$

3 Example: Quantization example: Harmonic Oscillator

Quantum for $f \in C^\infty(\mathbb{R} \times \mathbb{R})$

$$i\hbar \frac{d}{dt} f = -\frac{\hbar^2}{2} \frac{d^2 f}{dq^2} - \frac{q^2 f}{2}$$

Schrödinger's

Symmetries:

$$\Rightarrow \hat{a} = i\hbar \frac{d}{dq} + q \quad \hat{a}^\dagger = -i\hbar \frac{d}{dq} + q$$

$$[\hat{H}, \hat{a}] = i\hbar \hat{a} \quad [\hat{H}, \hat{a}^\dagger] = -i\hbar \hat{a}^\dagger$$

Let f_0 an eigenvector for $\hat{H} \Rightarrow \hat{H}(f_0) = \lambda f_0$

$$\Rightarrow \hat{H}(\hat{a} f_0) = (\lambda + i\hbar)(\hat{a} f_0) \quad \hat{a} f_0 \text{ \& \& } \hat{a}^\dagger f_0$$

$$\hat{H}(\hat{a}^\dagger f_0) = (\lambda - i\hbar)(\hat{a}^\dagger f_0) \quad \text{are eigenvectors}$$

$\hat{a}$ creation $\hat{a}^\dagger$ destruction

Hermite polynomials

Classic $\frac{d^2 q}{dt^2} = -q^2$ for $q \in C^\infty(\mathbb{R})$

Newton's

$$\text{Let } p := \frac{d}{dt} q \quad \Rightarrow \frac{dp}{dt} = -q^2$$

Hamilton-Jacobi eq for

$$H(p, q) = \frac{p^2}{2} - \frac{q^2}{2} \text{ in } M = \mathbb{R}^2$$

- This system have the symmetries

$$a(p, q) = p + q \quad \& \quad a^\dagger(p, q) = -p + q$$

$$\{H, a\} = p + q = a$$

$$\{H, a^\dagger\} = -p + q = -a^\dagger$$

3 Real Quantization (still using $\mathbb{C}$)

$$\text{Poisson Mfd } M \xleftarrow{\mathcal{Q}} \begin{matrix} \mathcal{A} \\ \cup \\ \mathcal{L} \\ \cap \\ \mathcal{N} \\ \cup \\ \mathcal{T} \end{matrix} \xrightarrow{\Delta} \begin{matrix} \text{A Hilbert space } L_M \\ \\ \text{an operator} \\ \hat{f} : \text{Dom}(\hat{f}) \rightarrow L_M \end{matrix}$$

$\forall f \in \mathcal{A} \subseteq C^\infty(M)$ _{Subalgebra}

such that

- $f \mapsto \hat{f}$ is $\mathbb{C}$ -linear
- $[\hat{f}, \hat{g}] = i\hbar \widehat{\{f, g\}} + \mathcal{O}(\hbar^2)$
- $\widehat{\left(\frac{\|p\|^2}{2} + V(q)\right)} = -\frac{\hbar^2}{2} \Delta + V$
- for P a power series in $\hbar$
 $P(\hbar) = \hat{P}(\hbar) + \mathcal{O}(\hbar)$

Example $M = \mathbb{R}^{2n}$

in $\mathbb{R}^n$ there is the Fourier transform for any $f \in C_c^\infty(\mathbb{R}^n)$ it is

$$\mathcal{F}(f)(p) = \int_{\mathbb{R}^n} e^{\frac{i}{\hbar} \langle p, q \rangle} f(q) dq$$

- it satisfy: $\mathcal{F}(\partial_q f)(p) = \frac{i}{\hbar} p_i \mathcal{F}(f)(p)$
- Therefore: $\mathcal{F}\left(-\frac{\hbar^2}{2} \Delta(f)\right) = \frac{\|p\|^2}{2} \mathcal{F}(f)(p)$
- this help us rewrite the Schrödinger operator as:

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$$= \iint e^{i\hbar \langle p, q - q' \rangle} H(p, q') f(q') dq' dp$$

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