Integració de foliacions singulars usant camins

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Outline:

- Intro to regular foliations
- Groupoids, holonomy, paths (Haefliger, Molino, Ehresmann)

Books

New

- Intro to singular foliations (Androulidakis, Skandalis, Debord)
- holonomy, paths for sing. foliations (A, S, D, Fernandes, Crainic, Villatoro, Zambon, G)

Regular Foliations

Infinitessimal definition

A regular foliation on a manifold M is a collection of vector fields $\mathcal{F} \subseteq \mathfrak{X}(M)$ such that for each point $p \in M$ there is some coordinates

$$(x_1, \dots, x_n) : \underset{\underset{M}{\cap}}{U} \longmapsto \mathbb{R}^n$$

with

$$\mathcal{F}|_U = \mathrm{Span}_{C^\infty(U)}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right)$$

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Global definition

A reg. fol. on a mfd. M is an equivalent relation $\sim$ (or a partition) s.t. $\forall\, p \in M$ $\exists$ coord $U \to \mathbb{R}^n$

$$q \mapsto (x_1(q), \dots, x_n(q))$$

where

$$\boxed{\begin{matrix} q_1 \sim q_2 \\ \text{in } U \end{matrix}} \iff \boxed{\begin{cases} x_{k+1}(q_1) = x_{k+1}(q_2) \\ \vdots \\ x_n(q_1) = x_n(q_2) \end{cases}}$$

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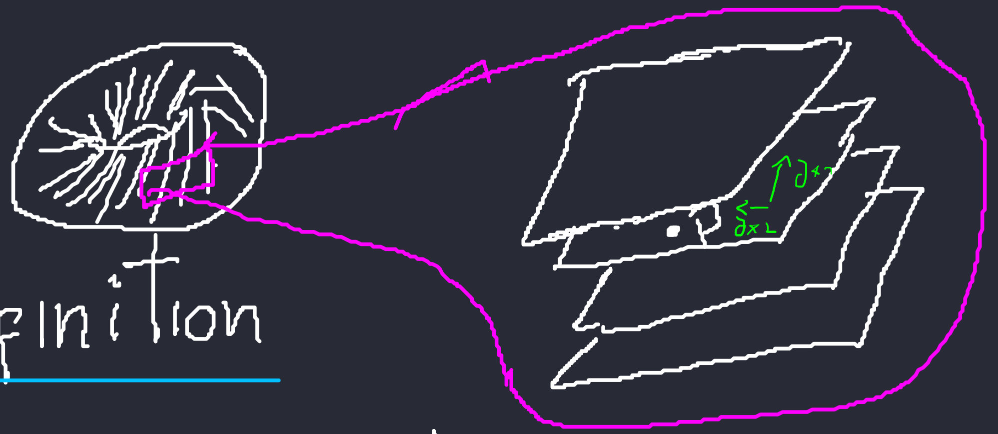

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Flow box theorem

The vector fields

$Y_1, \dots, Y_k \in \mathfrak{X}(M)$

are of the form $\frac{\partial}{\partial x_i}$ for some coordinates $(x_1, \dots, x_n)$ near $p \in M$

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$\exists$ nghd $U \subset M$ s.t.

a) $Y_1(q), \dots Y_k(q)$ are L.I. $\forall\, q \in U$

b) $[Y_i, Y_j]\big|_U = 0 \quad \forall\ 1 \leq j, i \leq k$

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Frobenius theorem

A regular foliation of dim $= k$

is equivalent to a collection of vector fields $\mathcal{F} \subseteq \mathfrak{X}(M)$

s.t.

a) $\mathcal{F}(p) = \{Y(p) : Y \in \mathcal{F}\} \subset T_pM$ is a subvector space of dim $= k$ $\forall\ p \in M$

b) $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$

Groupoids

is a small category where arrows are invertible, this is

- Set of $\boxed{\text{points } M}$ & set of $\boxed{\text{arrows } \mathcal{G}}$
- Maps $\boxed{s:\mathcal{G}\to M}$, $\boxed{t:\mathcal{G}\to M}$

$\boxed{\circ : \mathcal{G}\,{}_s\!\times_t \mathcal{G} \longrightarrow \mathcal{G}}$, $\boxed{e: M\to \mathcal{G}}$

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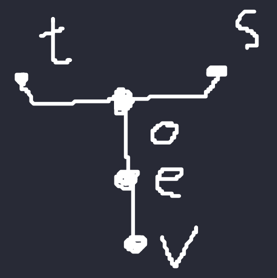

s.t.

a) $\circ$ is associative

b) for $y \xleftarrow{\varphi\in\mathcal{G}} x \in M$ $\quad e(y)\circ\varphi = \varphi\circ e(x) = \varphi$

c) $v(\varphi)\circ\varphi = e(x)$, $\quad \varphi\circ v(\varphi) = e(y)$

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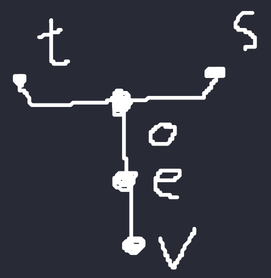

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c) $V(\varphi)\circ\varphi = e(x)$, $\varphi\circ V(\varphi) = e(y)$

Example

Let M be a set with an eq. rel. $\sim$

Take $\mathcal{G} = \text{Graph}(\sim) = \{(y,x)\in M\times M : y\sim x\}$

$\boxed{s(y,x) = x}$, $\boxed{t(y,x) = y}$

$\boxed{(z,y)\circ(y,x) = (z,x)}$

$\boxed{e(x) = (x,x)}$

$\boxed{V(y,x) = (x,y)}$

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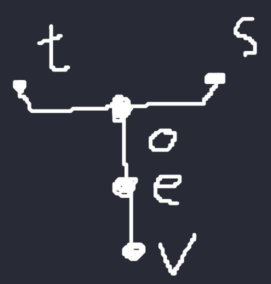

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$\boxed{s(y,x) = x}$, $\boxed{t(y,x) = y}$

$\boxed{(z,y) \circ (y,x) = (z,x)}$ (Transitivity)

$\boxed{e(x) = (x,x)}$ (Reflexive)

$\boxed{v(y,x) = (x,y)}$ (Simetric)

Lie Groupoids

is a small category where arrows are invertible, in mfd, this is

- mfd of $\boxed{\text{points } M}$ & mfd of $\boxed{\text{arrows } \mathcal{G}}$
- Maps $\boxed{s : \mathcal{G} \to M}$, $\boxed{t : \mathcal{G} \to M}$ ← Surjective Submersion

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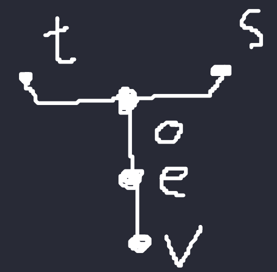
SMOOTH



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Example

Let M be a set with an eq. rel. $\sim$ smooth

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Example (Foliation local)

In $M = \mathbb{R}^k \times \mathbb{R}^{n-k}$ as points

Take $\mathcal{G} = \{\text{Paths constant in } \mathbb{R}^{n-k}\}/\text{Hom}$

$\boxed{s(\gamma) = \gamma(0)}$ $\boxed{t(\gamma) = \gamma(1)}$

$\circ$ concatenation

$e(x)$ constant path on x

$v(\gamma)$ inverse parametrization

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there are charts $\mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^{n-k} \xrightarrow{\approx} \mathcal{G}$ so it is smooth

$(x, y, c) \longmapsto (xt + (1-t)y, c)$

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For a mfd M with reg. fol $\mathcal{F}$ of dim k

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the same $s, t, \circ, e, v$ as before

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Singular Foliations on a mfd M (A.S.)

is a collection of v.f. $\mathcal{F} \subseteq \mathfrak{X}_c(M)$

s.t.

a) $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$ (closed under Lie brackets)

b) $\forall\, p \in M$ $\exists$ nghd $U \subseteq M$ & k_p

with $Y_1, \ldots, Y_{k_p} \in \mathfrak{X}(U)$

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Thm any sing. fol. gives an eq. rel $\sim_{\mathcal{F}}$

and each class $L_x = \{ y : y \sim_{\mathcal{F}} x \}$

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Results

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- Paths / Homotopy is a Lie groupoid

  $\Longleftrightarrow$ $\mathcal{F}$ is almost regular (AZ)

Smooth maps to a sing. fol. in a mfd M

a map $\mathcal{Y} : V_{\subseteq \mathbb{R}^n} \longrightarrow \mathcal{F}$ is smooth if

$\forall p \in M$ $\exists$ nghd $\mathcal{U} \in M$ with

$$\boxed{\mathcal{Y}(v)(q) = \sum_{i \in I} f_i(t, q) Y_i(q)}$$

$\forall q \in \mathcal{U}$, I finite, $Y_i \in \mathcal{F}$ & $f_i \in C^\infty(V \times M)$

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Paths to $\mathcal{F}$ is a collection of a

smooth map $Y : [0,1] \longrightarrow \mathcal{F}$

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a curve $\gamma : [0,1] \longrightarrow M$

s.t. $\dot{\gamma}(t) = Y(t, \gamma(t))$

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(Y_0, γ_0) & (Y_1, γ_1) is a smooth map

$Y : [0,1]^2 \longrightarrow \mathcal{F}$

$\gamma : [0,1]^2 \longrightarrow M$ $\quad \gamma(s,0) = \gamma_0(0) = \gamma_1(0)$

s.t.

- $Y(0,t,q) = Y_0(t,q)$ & $Y(1,t,q) = Y_1(t,q)$

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  $\gamma(0,t) = \gamma_0(t)$ & $\gamma(1,t) = \gamma_1(t)$
- $Y(s,t,q) = \frac{\partial}{\partial t} \gamma(t,t)$
- the vector field $W(s,t,q) = \frac{\partial}{\partial s} \Phi^{t,s}_Y$

  satisfy $W(s,1,\gamma(s,1)) \in I_{\gamma(s,1)} \mathcal{F}$ $\quad \forall s \in [0,1]$

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b) $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$

Example (General)

For a mfd M with reg. fol $\mathcal{F}$ of dim k

Take $\mathcal{G} = \{\text{Paths in a class}\} / \text{Homotopy of te class}$

the same s, t, o, e, v as before is a Lie Groupoid and

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$\boxed{10q}$

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Results

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- the almost smooth str. comes from a quotient of the mfd $\mathbb{R}^n \times U$ for U open in M (AS) & (GV)
- Paths / Homotopy on a leaf is smooth (Lie) (AZ)
- Paths / Homotopy is a Lie groupoid $\Leftrightarrow$ $\mathcal{F}$ is almost regular (AZ)

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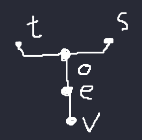

s.t.

a) o is associative

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Example

Let M be a set with an smooth eq. rel. $\sim$

Take $\mathcal{G} = \mathrm{Graph}(\sim) = \{(y, x) \in M \times M : y \sim x\}$

$\boxed{s(y, x) = x}$, $\boxed{t(y, x) = y}$

$\boxed{(z, y) \circ (y, x) = (z, x)}$ (Transitivity)

$\boxed{e(x) = (x, x)}$ (Reflexive)

$\boxed{v(y, x) = (x, y)}$ (Simetric)

Smooth maps to a sing. fol. in a mfd M

a map $Y : V_{\subseteq \mathbb{R}^n} \to \mathcal{F}$ is smooth if $\forall p \in M$ $\exists$ nghd $U \in M$ with

$\boxed{Y(v)(q) = \sum_{i \in I} f_i(t, q) Y_i(q)}$

$\forall q \in U$, I finite, $Y_i \in \mathcal{F}$ & $f_i \in C^\infty(V \times M)$

Paths to $\mathcal{F}$ is a collection of a

smooth map $Y : [0,1] \to \mathcal{F}$

&

a curve $\gamma : [0,1] \to M$

s.t. $\dot{\gamma}(t) = Y(t, \gamma(t))$

A $\mathcal{F}$-homotopy between paths in $\mathcal{F}$

(Y_0, γ_0) & (Y_1, γ_1) is a smooth map

$Y : [0,1]^2 \to \mathcal{F}$

$\gamma : [0,1]^2 \to M$ $\quad \gamma(s, 0) = \gamma_0(0) = \gamma_1(0)$

s.t.

- $Y(0, t, q) = Y_0(t, q)$ & $Y(1, t, q) = Y_1(t, q)$
  $\gamma(0, t) = \gamma_0(t)$ & $\gamma(1, t) = \gamma_1(t)$
- $Y(s, t, q) = \frac{\partial}{\partial t} \gamma(t, t)$
- the vector field $W(s, t, q) = \frac{\partial}{\partial s} \Phi_Y^{t,s}$ satisfy $W(s, 1, \gamma(s, 1)) \in I_{\gamma(s,1)} \mathcal{F}$ $\forall s \in [0, 1]$