# PB groupoids & bundle gerbes

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# Outline

① Lie 2-groups & examples

② PB groupoids

③ Bundle gerbes & PB groupoids

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② PB groupoids

③ Bundle gerbes & PB groupoids

## Lie 2-group

Homomorphisms

$$G^{(1)} \rightrightarrows G$$

Lie groups

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1. Lie 2-groups & examples
2. PB groupoids
3. Bundle gerbes & PB groupoids

**Lie 2-group**

Homomorphisms

$$G^{(1)} \rightrightarrows G$$

Lie groups

**PB groupoid**

$$\boxed{G^{(1)} \curvearrowright P^{(1)} \longrightarrow M^{(1)}}$$

$$\boxed{\downarrow \quad \downarrow}$$ Pb morphisms

$$\boxed{G \curvearrowright P \longrightarrow M}$$

Principal bundles

# Outline

1. Lie 2-groups & examples

2. PB groupoids

3. Bundle gerbes & PB groupoids

## Lie 2-group

Homomorphisms

$$\begin{tikzcd} G^{(1)} \arrow[r, shift left] \arrow[r, shift right] & G \end{tikzcd}$$

Lie groups

## PB groupoid

$$\begin{tikzcd} G^{(1)} \curvearrowright P^{(1)} \arrow[d] \arrow[r] & M^{(1)} \arrow[d] \\ G \curvearrowright P \arrow[r] & M \end{tikzcd}$$

Pb morphisms

Principal bundles

## Bundle gerbe

$$\begin{tikzcd} G^{(1)} \arrow[d, shift left] \arrow[d, shift right] & \curvearrowright B \arrow[dl] \arrow[r] \arrow[dr, shift left] \arrow[dr, shift right] & M^{(1)} \arrow[d, shift left] \arrow[d, shift right] \\ G & & M \end{tikzcd}$$

$PG^{(1)}$-bundle (groupoid action)

Lie groupoids

# ① Lie 2-Groups

$G^{(1)} \rightrightarrows G \Rightarrow$ $G^{(1)} \underset{N}{\overset{S}{\rightleftarrows}} G$

Lie groups

Homomorphism

$\Rightarrow$

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$\Rightarrow$ Algebra result $\boxed{G^{(1)} \approx \mathrm{Ker}(s) \rtimes G}$

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$$(h, g) \longmapsto h \cdot N(g)$$
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Call $H := \mathrm{Ker}(s)$ is a Lie group

$d = t|_H : H \to G$ is homomrf

$H \curvearrowleft G \quad h \cdot g = d(h) \cdot g \Rightarrow \boxed{H \times G \rightrightarrows G \text{ Lie grd}}$

Action grpd

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Action grpd

**Prop** Any Lie 2-grp is equivalent to a "Crossed Module". i.e.

a) $d : H \to G$ a homomrf

b) $C : G \to \mathrm{Aut}(H)$ (usually omitted)

s.t. $\forall$ $g \in G$ $h \in H$ $h' \in H$

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**Proof** (⇒)

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For a Crossed Mod $(G, H, d, C)$

$(H \times G, \rtimes) \rightrightarrows (G, \cdot)$

the action grpd of $H$ in $G$ is a Lie 2-grp

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# Examples

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2. identity $G \rightrightarrows G$

3. $H$ normal subgroup of $G$ $\quad H \rtimes G \rightrightarrows G$

4. given a morph $G \to Gl(V)$

the canonical map $V \xrightarrow{d} G$, $v \mapsto e_G$

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6. $Pin(V) \xrightarrow{p} O(V)$, $O(V) \to Aut(Pin(V))$
   is a crossed module
   $Pin(V)\rtimes O(V) \rightrightarrows O(V)$ is a Lie 2-group

## ② PB groupoids

$$\mathcal{G}^{(1)} \curvearrowright P^{(1)} \longrightarrow M^{(1)}$$

Lie 2-group action

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Def A Lie 2-group action is a Lie grp act

$$(H \rtimes G) \times P^{(1)} \xrightarrow{a} P^{(1)} \qquad \text{s.t.}$$

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This implies that

- $G = \{e\} \rtimes G$ acts by Lie grpd mph
- the map $\varphi : H \times P \to P^{(1)}$, $(h,p) \mapsto (h,e) \cdot N(p)$ satisfies

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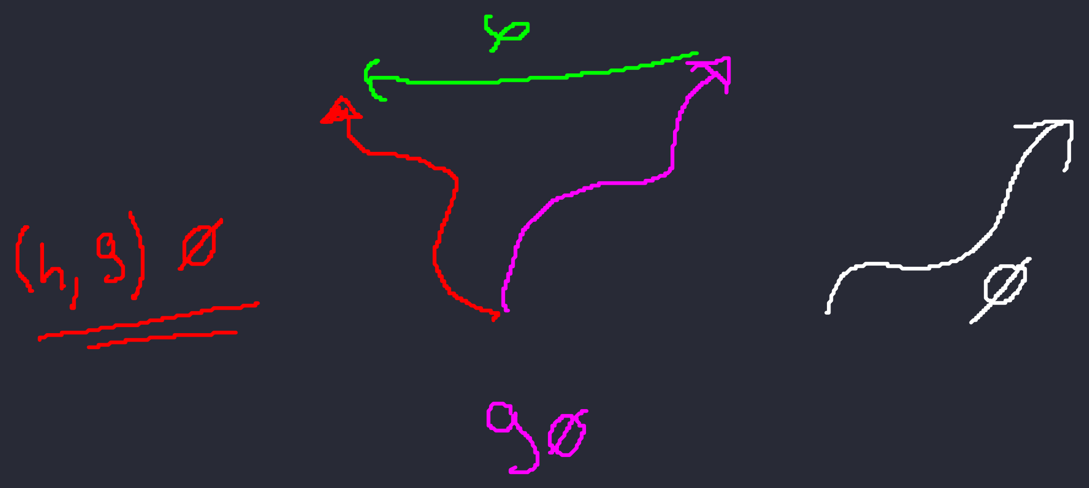

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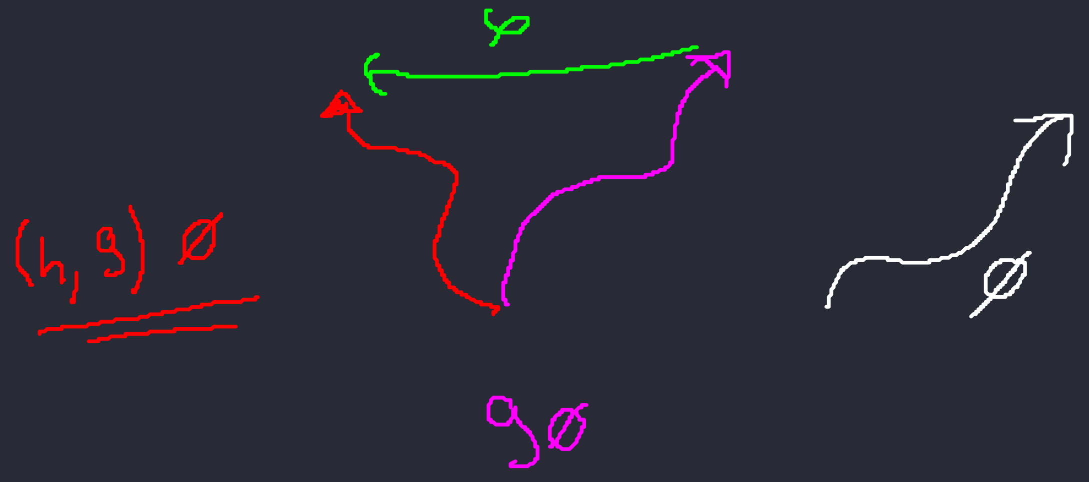


**Prop** if $(H \rtimes G) \curvearrowright P^{(1)}$ is free and proper

$\Rightarrow P^{(1)}/(H \rtimes G)$ is a Lie grpd over $P/G$

## Examples

Given $G \curvearrowright P \xrightarrow{\pi} M$ a principal bundle

a) The identity grpds generate a PB grpd

b) Pair grpds $G^2 \curvearrowright P^{(2)} \to M^{(1)}$ is a PB grpd

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denote $P^{(1)} = \pi^{-1} M^{(1)} = P \,{}_{\pi}\!\times_{t} M^{(1)}_{s} \times_{\pi} P$

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This last one is related to $A_M$-con and Atiyah sequences

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d) given any surjective vector bundle map

$W \xrightarrow{\phi} E$ over a mfd $M$

$\Rightarrow$ $V = \ker(\phi)$ is a v.b.

take $P = \mathrm{Frames}(V) \xrightarrow{\pi} M$

$\Rightarrow$ short exact segence

$\mathbb{R}^k \times P \simeq \pi^* V \longrightarrow \pi^* W \to \pi^* E$

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$$\mathbb{R}^k \times P \simeq \pi^*V \longrightarrow \pi^*W \to \pi^*E$$

$\Rightarrow$

$$\boxed{\begin{tikzcd} \mathbb{R}^k \rtimes Gl(k) \arrow[r, hook] \arrow[d, shift left] \arrow[d, shift right] & \pi^*W \arrow[r] \arrow[d, "s=t"] & E \arrow[d, "s=t"] \\ Gl(k) \arrow[r, hook] & P \arrow[r] & M \end{tikzcd}}$$

# ③ Bundle gerbes & PB grpds

A bundle gerbe is

$$\begin{tikzcd}
H\rtimes G \arrow[d, shift left] \arrow[d, shift right] & \curvearrowright B^{(1)} \arrow[r] \arrow[dl, "\rho"] \arrow[d, shift left] \arrow[d, shift right] & M^{(1)} \arrow[dl, shift left] \arrow[dl, shift right] \\
G & M &
\end{tikzcd}$$

- Principal (groupoid-action) bundle
- grpd mrph

s.t. $\forall \; k_i \in H\rtimes G \quad b_i \in B$

$\Rightarrow (k_1 * b_1)\circ(k_2 * b_2) = (k_1 \cdot k_2) * (b_1, b_2)$

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a PB grpd

$S(g,b) = (g, S(b))$ $\qquad t(g,b) = (g\rho(b), t(b))$

$(h,g)\cdot(g',b) = \left(gg', (c_{\rho(b)}\, g'^{-1} g^{-1} h, \rho(b)) * b\right)$

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Inverse functor

$$\begin{array}{ccccc} H\rtimes G & \curvearrowright & P^{(1)} & \longrightarrow & M^{(1)} \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ G & \curvearrowright & G\times M & \longrightarrow & M \end{array}$$

$\Rightarrow P^{(1)} \approx G \times \underset{\displaystyle B}{\underset{\shortparallel}{\mathrm{Ker}(pr_G \circ s)}}$ diffeo

$\Rightarrow B$ aquires a groupoid str $P^{(1)}/G$

$\Rightarrow \boxed{P^{(1)} \approx G\times B}$

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A bundle gerbe is

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It works for morphisms

$\mathrm{Mor}(P^{(1)}) = (H\rtimes G)$ equivariant Maps $P^{(1)} \to H\rtimes G$

$\Updownarrow$

$\mathrm{Mor}(B) = H\rtimes G$ equivariant Maps $P^{(1)} \to H$

# 10q

## ① Lie 2-Groups

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Lie groups — Homomorphism

$\Rightarrow$ Algebra result $\boxed{G^{(1)} \approx \mathrm{Ker}(S) \rtimes G}$

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the map $H \times G \to G^{(1)}$
$(h, g) \mapsto h \cdot N(g)$

is a Lie 2-group isomorphism (Lie group & Lie groupoid iso)

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b) $C : G \to \mathrm{Aut}(H)$ (usually omitted)
s.t. $\forall\ g \in G \quad h \in H \quad h' \in H$
$d(C_g h) = g\, d(h)\, g^{-1}$ & $C_{dh} h' = h h' h^{-1}$

Proof ($\Rightarrow$)
$G^{(1)} \rightrightarrows G$ a Lie 2grp
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$(H \times G, \rtimes) \rightrightarrows (G, \cdot)$
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## Examples

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① Pair groupoid $G \times G \rightrightarrows G$

$\mathrm{ker}(S) = G \times \{e\} \approx G$
$N : G \to G \times G, \quad g \mapsto (g, g)$

- $G \rtimes G \approx G \times G$
$(h, g) \mapsto (h, e) \cdot (g, g) = (hg, g)$

- $G \rtimes G$ is action groupoid of $G$ on $G$
$G \times G$ is the pair groupoid

② identity $G \rightrightarrows G$

③ $H$ normal subgroup of $G$ $\quad H \rtimes G \rightrightarrows G$

④ given a mrph $G \to Gl(V)$
the canonical map $V \xrightarrow{d} G,\ v \mapsto e_G$
induce a crossed module so
$V \rtimes G \rightrightarrows G$ is a Lie 2-group

⑤ $TG \approx \mathfrak{g} \rtimes G$ is a Lie 2-group

⑥ $\mathrm{Pin}(V) \xrightarrow{P} O(V), \quad O(V) \to \mathrm{Aut}(\mathrm{Pin}(V))$
is a crossed module
$\mathrm{Pin}(V) \rtimes O(V) \rightrightarrows O(V)$ is a Lie 2-group

# thank you!

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## ② PB groupoids

$G^{(1)} \curvearrowright P^{(1)} \to M^{(1)}$

Lie 2-group action

Def A Lie 2-group action is a Lie grp act

$(H \rtimes G) \times P^{(1)} \xrightarrow{a} P^{(1)}$ s.t.

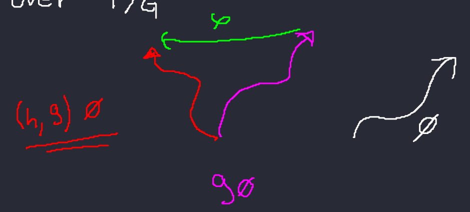
"a" is a Lie grpd mph i.e.

This implies that

- $G = \{e\} \rtimes G$ acts by Lie grpd mph
- the map $\varphi : H \times P \to P^{(1)},\ (h, p) \mapsto (h, e) \cdot N(p)$ satisfies
$(h, g)\,\phi = \varphi(h, g\,t(\phi)) \circ (g\phi)$

Prop if $(H \rtimes G) \curvearrowright P^{(1)}$ is free and proper

$\Rightarrow P^{(1)} / (H \rtimes G)$ is a Lie grpd over $P/G$




## Examples

Given $G \curvearrowright P \xrightarrow{\pi} M$ a principal bundle

a) The identity grpds generate a PBgrpd

b) Pair grpds $G^2 \curvearrowright P^{(2)} \to M^{(1)}$ is a PB grpd

c) Given $M^{(1)} \rightrightarrows M$ any Lie grpd
denote $P^{(1)} = \pi^{-1} M^{(1)} = P \,{}_\pi\!\times_t M^{(1)} {}_s\!\times_\pi P$

$\Rightarrow \boxed{(G \rtimes G) \curvearrowright P^{(1)} \to M^{(1)} \text{ is a PB grpd}}$

and its derivative $\boxed{TG \curvearrowright A_P \to A_M \text{ is a PBgrpd}}$

This last one is related to $A_i$-con and Atiyah sequences

d) given any surjective vector bundle map
$W \xrightarrow{\varnothing} E$ over a mfd $M$

$\Rightarrow V = \ker(\varnothing)$ is a v.b.

take $P = \mathrm{Frames}(V) \xrightarrow{\pi} M$

$\Rightarrow$ short exact sequence
$\mathbb{R}^k \times P \approx \pi^* V \to \pi^* W \to \pi^* E$

$\Rightarrow$
$$\boxed{\begin{array}{ccccc} \mathbb{R}^k \rtimes Gl(k) & \curvearrowright & \pi^* W & \longrightarrow & E \\ \downdownarrows & & \downdownarrows s{,}t & & \downdownarrows s{=}t \\ Gl(k) & \curvearrowright & P & \longrightarrow & M \end{array}}$$

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## ③ Bundle gerbes & PB grpds

A bundle gerbe is

$$\begin{array}{ccccc} H \rtimes G & \curvearrowright & B^{(1)} & \to & M^{(1)} \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ G & & & & M \end{array}$$

- Principal (groupoid-action) bundle
- grpd mrph

s.t. $\forall\ k_i \in H \rtimes G \quad b_i \in B$

$\Rightarrow (k_1 * b_1) \circ (k_2 * b_2) = (k_1 \cdot k_2) * (b_1, b_2)$

$$\boxed{\begin{array}{ccccc} H \rtimes G & \curvearrowright & G \times B & \longrightarrow & M^{(1)} \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ G & \curvearrowright & G \times M & \to & M \end{array}}$$ a PB grpd

$S(g, b) = (g, S(b)) \qquad t(g, b) = (g t(b), t(b))$

$(h, g) * (g', b) = (gg', (C_{(h,g)} g^{-1} g'^{-1} h, b) * b)$

$$\begin{array}{ccccc} H \rtimes G & \curvearrowright & P^{(1)} & \to & M^{(1)} \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ G & \curvearrowright & G \times M & \longrightarrow & M \end{array}$$

Inverse functor

$\Rightarrow P^{(1)} \approx G \times \underbrace{\mathrm{Ker}(\mathrm{Pr}_G \circ S)}_{= B}$ diffeo

$\Rightarrow B$ aquires a groupoid str $P^{(1)}/G$

$\Rightarrow \boxed{P^{(1)} \approx G \times B}$

IT works for morphisms

$\mathrm{Mor}(P^{(1)}) = (H \rtimes G)$ equivariant Maps $P^{(1)} \to H \rtimes G$

$\mathrm{Mor}(B) = H \rtimes G$ equivariant Maps $P^{(1)} \to H$

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